Exercise Sheet: Introduction to AdS/CFT

Exercise 1: String actions

In lecture we introduced the Polyakov action for the string:

$$S_P = \frac{1}{4\pi l_s^2} \int d\sigma d\tau \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \, .$$

a. Substitute the equation of motion for the world-sheet metric $g_{\alpha\beta}$ in the action and show that S_P is classically equivalent to the Nambu-Goto action

$$S_{NG} \propto \int d\sigma d\tau \sqrt{\det \partial_\alpha X^\mu \partial_\beta X_\mu} \, .$$

b. Show that this is just the area of a string that it sweeps as it moves in time.

c. Work out the variational principle for the Polyakov action above and show that there are essentially three different types of solutions below. Use Dirichlet boundary conditions in time as usual: $\delta X^\mu(\tau_0) = \delta X^\mu(\tau_1) = 0$ for initial and final times.

1. Closed string: $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$. Length of the closed string is chosen as 2π for simplicity.
2. Open string: $\partial_\sigma X^\mu(0, \tau) = \partial_\sigma X^\mu(\pi, \tau) = 0$. Length of the open string is chosen as π for simplicity.
3. Open string on a Dp brane: $\partial_\sigma X^\mu(0, \tau) = \partial_\sigma X^\mu(\pi, \tau) = 0$ for $\mu = 0, 1, \cdots p$. $\delta X^\mu(0, \tau) = \delta X^\mu(\pi, \tau) = 0$ for $\mu = p+1, p+2, \cdots D$.

Exercise 2: String modes and energy-momentum

a. Show that the Polyakov action is invariant under three local symmetries: The two diffeomorphisms $\tau^\alpha \to \tau^\alpha + \xi^\alpha(\sigma, \tau)$, $g_{\alpha\beta} \to g_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$, for $\tau^0 = \tau$, $\tau^1 = \sigma$. And the Weyl symmetry: $g_{\alpha\beta} \to e^{2\omega(\sigma,\tau)} g_{\alpha\beta}$.
b. Show that this freedom can be used to fix $g_{\alpha\beta} = \eta_{\alpha\beta}$, at least locally at every point (σ, τ).
c. Work out the equation of motions for the X^μ fields in flat space time and in the gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ and show that

$$X^\mu(\sigma, \tau) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma), \qquad \text{(open string)}$$

$$X^\mu(\sigma, \tau) = x^\mu + \frac{p^\mu}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} e^{-in\tau} \left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma} \right). \quad \text{(closed string)}$$

d. Show that the components of the world-sheet energy-momentum tensor in the conformal gauge ($g_{\alpha\beta} = \eta_{\alpha\beta}$) read

$$T_{01} = T_{10} = \frac{1}{2} \dot{X} \cdot X', \qquad T_{00} = T_{11} = \frac{1}{4} \left(\dot{X}^2 + X'^2 \right) .$$

e. Obtain T_{00} in terms of α oscillators both for the closed and the open string.

Exercise 3: Mass spectrum of the string

a. Show that the equation of motion for the world-sheet metric $g_{\alpha\beta}$ requires vanishing of the world-sheet energy momentum tensor $T_{\alpha\beta} = 0$ classically.
b. In quantum theory this cannot be maintained as there is a normal ordering anomaly in the product of α operators. The normal ordering is defined as $: \alpha_n^\mu \alpha_m^\nu := \alpha_n^\mu \alpha_m^\nu$ for $m < 0$, $n > 0$ and $: \alpha_n^\mu \alpha_m^\nu := \alpha_m^\nu \alpha_n^\mu$ for $m > 0$, $n < 0$. In quantum mechanics the condition (Hamiltonian constraint) $T_{00} = 0$ becomes

$$(\hat{T}_{00} - 1)|\psi\rangle = 0\,,$$

where the -1 bit is due to normal ordering anomaly (think of the 1/2 in the harmonic oscillator).
Show that this condition yields the mass formula

$$m^2 = \frac{1}{l_s^2}\left(\sum_{m>0} \alpha_{-m}\cdot\alpha_m - 1\right),$$

for the open string and

$$m^2 = \frac{2}{l_s^2}\left(\sum_{m>0} (\alpha_{-m}\cdot\alpha_m + \bar{\alpha}_{-m}\cdot\bar{\alpha}_m - 2)\right),$$

for the closed string.
c. Show that the massless open string state is a gauge field. Similarly show that the massless closed string state includes a scalar field Φ, the dilation, a traceless symmetric tensor $h_{\mu\nu}$, that is the graviton, and an antisymmetric tensor $B_{\mu\nu}$.

Exercise 4: Scalar field in AdS

In class we wrote down a metric for the AdS_5 space-time

$$ds^2 = \frac{dr^2}{r^2} + r^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\,.$$

Another useful coordinate system is obtained by $z = 1/r$.
a. Show that in this coordinate system the metric becomes

$$ds^2 = \frac{1}{z^2}(dz^2 - dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\,.$$

b. Derive the equation of motion for a scalar field Φ in this space with mass m^2.

c. Show that near the boundary $z = 0$

$$\Phi(z, x) \to \phi_-(x) z^{\Delta_-} + \phi_+(x) z^{\Delta_+} + \cdots$$

where

$$\Delta_-\Delta_+ = -m^2, \qquad \Delta_- + \Delta_+ = 4\,.$$

Exercise 5: Bulk to boundary propagator in AdS

Show that

$$K_\Delta(z,\vec{x},\vec{y}) = \frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}\left(\frac{z}{z^2+(\vec{x}-\vec{y})^2}\right)^\Delta$$

is a solution to the scalar equation of motion with the boundary condition

$$z^{\Delta-4}K_\Delta(z,\vec{x},\vec{y}) = \delta(\vec{x},\vec{y})\,,$$

as $z\to 0$ and

$$K_\Delta(z,\vec{x},\vec{y}) \to 0\,,$$

as $z\to\infty$ (regularity in the interior).

Exercise 6: Spectrum of a dimension 4 operator in dual field theory

a. Fourier transform the scalar field $\Phi(z,\vec{x})$ only in the Minkowski directions $\vec{x}$, not z, and obtain the equation of motion for a scalar field that is dual to a dimension 4 operator as

$$\Phi''(z,k) - \frac{3}{z}\Phi'(z,k) - k^2\Phi(z,k) = 0\,.$$

b. We will now obtain the mass spectrum of states created by this operator acting on the vacuum as $m_4^2 = -k^2$. To obtain a gaped spectrum, let us assume there is an IR cut-off at $z = z_0$ where the AdS geometry terminates. Show that requiring the boundary conditions (i) normalizability near $z = 0$ and (ii) regularity $\Phi(z_0) = 0$ at the IR cut-off determines the spectrum as

$$J_2(m_4 z_0) = 0,$$

where J_2 is a Bessel function. What are the first few states?

Exercise 7: AdS-Schwarzchild black-hole

Show that the Einstein's equation that follow from the action

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\,(R+\Lambda)$$

has a solution of the form

$$ds^2 = \frac{1}{z^2}(\frac{dz^2}{f(z)} - f(z)dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\,,$$

with $f(z) = 1-(z/z_h)^4$.