Exercises Cosmology

March 2019

1. The 3-D unit sphere can be embedded in 4 dimensional flat space and is defined by the equation
$$x^2 + y^2 + z^2 + w^2 = 1$$
The line element in 4-D flat space is
$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$
a. Define the 3-D notation $\mathbf{r} = (x, y, z)$ and derive the line element measuring local distances between points on the sphere:
$$ds^2 = d\mathbf{r}^2 + \frac{\mathbf{r} \cdot d\mathbf{r}}{1 - \mathbf{r}^2}$$
b. Show that in 3-D polar co-ordinates
$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta$$
the line-element on the sphere can be rewritten as
$$ds^2 = \frac{dr^2}{1 - r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$
c. Do the same exercise for the 3-D unit hyperboloid defined by
$$x^2 + y^2 + z^2 - w^2 = -1$$
and derive that in 3-D polar co-ordinates
$$ds^2 = \frac{dr^2}{1 + r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

2. The fundamental equations of cosmology for a homogeneous and isotropic universe are
$$\frac{d(\varepsilon a^3)}{dt} + p\frac{da^3}{dt} = 0, \qquad H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\varepsilon}{3} - \frac{k}{a^2} \tag{1}$$
with $k = 0, \pm 1$.
a. Take an equation of state of the form
$$p = w\varepsilon,$$
with w a constant. show that
$$\varepsilon a^{3(1+w)} = \text{constant}$$

b. In a spatially flat universe ($k = 0$), derive that for $w \neq -1$:

$$a(t) = \text{constant} \times t^{2/[3(1+w)]}$$

provided we take the initial condition $a(0) = 0$; and for $w = -1$:

$$a(t) = \text{constant} \times e^{Ht}$$

3. From eqn. (1) and the definition of the critical energy density

$$\varepsilon_c \equiv \frac{3H^2}{8\pi G}$$

show that a universe filled with radiation, cold matter an vacuum energy has an energy balance that can be written as

$$\frac{\varepsilon_c}{\varepsilon_{c0}} = \frac{H^2}{H_0^2} = \Omega_r x^{-4} + \Omega_m x^{-3} + \Omega_k x^{-2} + \Omega_v,$$

where $x = a/a_0$ and

$$\Omega_r = \frac{\varepsilon_{r0}}{\varepsilon_{c0}}, \quad \Omega_m = \frac{\varepsilon_{m0}}{\varepsilon_{c0}}, \quad \Omega_c = \frac{\varepsilon_{v0}}{\varepsilon_{c0}}, \quad \Omega_k = -\frac{k}{a_0^2 H_0^2}$$