BMS algebra and its susy extensions in 3D

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General relativity - Geometry and physics and their subtle interplay

In General relativity physics and geometry coexist in a delicate equilibrium. The dynamical aspects of the metric are at the center of the profound physical results of the theory (e.g. Black holes, cosmology, ...), but at the same time they are an enormous complication (non-linearity, **boundary conditions**, ...)

Think of radiation: the fall- off conditions for the fields (which in flat spacetime are easily and intuitively expressed as subleading terms in a $1/r$ expansion for $r \to \infty$) in general relativity are quite subtle, since the entity which gives meaning to them is the metric itself, now a dynamical variable.

It took 40 yrs from the formulation of GR before the boundary conditions at spacelike infinity (massive particles) were understood and definitions were given for the conserved charges of an isolated system!

Gravitational radiation and BMS symmetries

Gravitational radiation adds another layer of complexity.
It was only in the 60's that this long standing problem was resolved by approaching it in a invariant way, namely move the analysis to the natural frame for massless particles: **null infinity**.

This invariant description gave birth to what are now known as **BMS symmetries**, i.e. transformations which preserve the boundary conditions of the metric, but are NOT isometries!

Bondi-van der Burg-Metzner-Sachs (BMS) symmetries

The group of symmetries at null infinity of spacetimes which asymptote flat Minkowski is much larger than the Poincare' group.

It is the semi-direct of the Lorentz transformations and an infinite dimensional abelian group of 'supertranslations', i.e. translation along the null coordinate, which contains the physical asymptotic translations.

If one allows for local singularities in the asymptotic symmetry generators, then also the Lorentz group extends to another infinite dimensional group, 'superrotations'.

Outline of the talk

- Basic of BMS_4
- Applications: the Strominger triangle + information paradox

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- Basic of BMS_4
- Applications: the Strominger triangle + information paradox
- BMS_3 and its supersymmetrizations
- Novel susy algebra with exotic energy bounds

The metric ansatz

Asymptotically flat solutions with coordinates $x^\mu = (u, r, x^A)$ can be written in terms of 7 unknown functions: V, β, g_{AB}, U^A as follows:

$$g_{\mu\nu} = \begin{pmatrix} \frac{V}{r} e^{2\beta} + g_{AB} U^A U^B & e^{2\beta} & -g_{BC} U^C \\ e^{2\beta} & 0 & 0 \\ -g_{AC} U^C & 0 & g_{AB} \end{pmatrix}$$

or equivalently :

$$\mathrm{d}s^2 = \frac{V}{r} e^{2\beta} \mathrm{d}u^2 + 2e^{2\beta} \mathrm{d}u \mathrm{d}r + g_{AB}(U^A \mathrm{d}u - \mathrm{d}x^A)(U^B \mathrm{d}u - \mathrm{d}x^B)$$

with boundary conditions

$$\beta = \mathcal{O}(r^{-2}) \, , \; U^A = \mathcal{O}(r^{-2}) \, , \; g_{AB} = r^2 \gamma_{AB} + \mathcal{O}(r) \, ,$$

$$\det(g_{AB}) = r^4 \det(\gamma_{AB}) \, , \; \frac{V}{r} = -1 + \mathcal{O}(r^{-1})$$

$$\gamma_{AB} \mathrm{d}x^A \mathrm{d}x^B = \mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\phi^2 = \frac{4}{(1 + z\bar{z})^2} \mathrm{d}z \mathrm{d}\bar{z}$$

Symmetries of asymptotically flat space

Those are gauge symmetries that preserve boundary conditions, i.e.

$$\mathcal{L}_\xi g_{\mu\nu} = \begin{pmatrix} \mathcal{O}(r^{-1}) & \mathcal{O}(r^{-2}) & \mathcal{O}(1) \\ \mathcal{O}(r^{-2}) & 0 & 0 \\ \mathcal{O}(1) & 0 & \mathcal{O}(r) \end{pmatrix}$$

$$\mathcal{L}_\xi(\det(g_{AB})) = \det(g_{AB})\, g^{CD}\mathcal{L}_\xi g_{CD} = 0$$

imposed to ensure the Riemann tensor falls-off sufficiently fast.

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imposed to ensure the Riemann tensor falls-off sufficiently fast.
Let us solve, off-shell, for this Killing vector preserving the boundary conditions, in two steps:

$$\mathbf{1})\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{rA} = g^{CD}\mathcal{L}_\xi g_{CD} = 0$$
$$\mathbf{2})\mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2})\,, \mathcal{L}_\xi g_{uA} = \mathcal{O}(1)\,, \mathcal{L}_\xi g_{AB} = \mathcal{O}(r)\,, \mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1})$$

Solve conformal Killing vector equation

$$
\begin{aligned}
&\mathbf{1)}\mathcal{L}_\xi g_{rr} = 0 \Rightarrow 2\nabla_r \xi_r = 2g_{ur}\nabla_r \xi^u = 2g_{ur}\partial_r \xi^u = 0 \Rightarrow \xi^u = f(u, x^A)\ ,\\
&\quad \mathcal{L}_\xi g_{rA} = 0 \Rightarrow \nabla_{(r}\xi_{A)} = 0 \Rightarrow \xi^A = Y^A(u, x^B) - \partial_B f \int_r^\infty \mathrm{d}r' e^{2\beta} g^{AB}\ ,\\
&\quad g^{CD}\mathcal{L}_\xi g_{CD} = 0 \Rightarrow \xi^r = -\frac{r}{2}(\mathcal{D}_A Y^A - U^c \partial_C f)\ ,\\
&\mathbf{2)}\mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}) \Rightarrow \partial_u f = \tfrac{1}{2}\mathcal{D}_A Y^A \Rightarrow \xi^u = T(x^A) + \frac{u}{2}\mathcal{D}_A Y^A\ ,\\
&\quad \mathcal{L}_\xi g_{uA} = \mathcal{O}(r^{-2}) \Rightarrow \partial_u Y^A = 0 \Rightarrow Y^A = Y^A(x^B)\ ,\\
&\quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r) \Rightarrow \text{conformal Killing equation for } Y^A\ ,\\
&\quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r) \Rightarrow \text{identity for conformal Killing vectors}\ .
\end{aligned}
$$

Solve conformal Killing vector equation

The symmetries preserving the boundary conditions are generated by:

$$\xi^u = T(x^A) + \frac{u}{2}\mathcal{D}_A Y^A$$

$$\xi^A = Y^A(x^A) - \partial_B f \int_r^\infty \mathrm{d}r' e^{2\beta} g^{AB}$$

$$\xi^r = -\frac{r}{2}(\mathcal{D}_A Y^A - U^c \partial_C f)$$

Expanding in powers of inverse r and at asymptotic null infinity $r \to \infty$:

$$\xi = T(x^A)\partial_u + \left(\frac{u}{2}\mathcal{D}_A Y^A \partial_u + Y^A(x^B)\partial_A\right)$$

IMPORTANT: if r is finite, there is still infinitely many symmetry generators! They stem from the null frame, so the same symmetries arise on any null surface, e.g. black hole horizons!

Old approach - Global BMS_4

$$\xi = T(x^A)\partial_u + \left(\frac{u}{2}\mathcal{D}_A Y^A \partial_u + Y^A(x^B)\partial_A\right)$$

The functions T^A and Y^A are globally well-defined and can be developed in spherical harmonic functions on the sphere.

- T^A are smooth functions on the sphere - ∞ supertranslations
- Y^A are global conformal Killing vector of the 2-sphere - Lorentz transformations

New approach - Local BMS_4

$$\xi = T(x^A)\partial_u + \left(\frac{u}{2}\mathcal{D}_A Y^A \partial_u + Y^A(x^B)\partial_A\right)$$

The functions T^A and Y^A are meromorphic functions (admit pole singularities) and are developed in Laurent series.

New approach - Local BMS$_4$

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Use conformally flat metric on the 2-sphere :

$$z = e^{\mathrm{i}\phi}\cot\frac{\theta}{2}\ , \qquad \bar{z} = e^{-\mathrm{i}\phi}\cot\frac{\theta}{2}$$

$$\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{4}{(1+z\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z}$$

$$Y^z(z) = Y\ , \qquad Y^{\bar{z}}(\bar{z}) = \bar{Y}$$

then the asymptotic conformal Killing vector reads:

$$\xi_{T,Y,\bar{Y}} = T(z,\bar{z})\partial_u + \left(\frac{u}{2}(\partial_z Y + \partial_{\bar{z}}\bar{Y}) - \frac{u}{1+z\bar{z}}(\bar{z}Y + z\bar{Y})\right)\partial_u + Y\partial_z + \bar{Y}\partial_{\bar{z}}$$

Algebra for local BMS_4

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If we expand the generators as follows:

$$\begin{aligned} l_n &= \{\xi_{T,Y,\bar{Y}} : T = 0,\, Y = -z^{n+1},\, \bar{Y} = 0\}, \\ \bar{l}_n &= \{\xi_{T,Y,\bar{Y}} : T = 0,\, Y = 0,\, \bar{Y} = -\bar{z}^{n+1}\}, \\ t_{n,m} &= \{\xi_{T,Y,\bar{Y}} : T = \frac{z^n \bar{z}^m}{1+z\bar{z}},\, Y = 0,\, \bar{Y} = 0\}\ , \end{aligned}$$

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we can write down the algebra:

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m}\ , \qquad & [l_p, t_{n,m}] &= \left(\frac{p+1}{2} - n\right)t_{n+p,m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m}\ , \qquad & [\bar{l}_p, t_{n,m}] &= \left(\frac{p+1}{2} - m\right)t_{n,m+p} \end{aligned}$$

Integrating the Einstein's equation

So far we've used the generic ansatz for the metric of asymptotically flat spaces to derive off-shell results. One can push it further by integrate the Bianchi identities and the equations of motion of the metric.

$$\nabla_{[\alpha} R_{\mu\nu]\rho\sigma} = 0 \Rightarrow \nabla_\alpha \left(R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R \right) = 0 \, ,$$
$$R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R = 0$$

while using the following expansions in powers of r (INITIAL DATA):

$$g_{AB} = r^2 \, \gamma_{AB} + r \, C_{AB} + \mathcal{O}(1) + \mathcal{O}(r^{-1}) + \dots$$

Solution and initial-data equations

We find the functions:

$$\beta = -\frac{1}{32}\frac{C^A{}_B C^B{}_A}{r^2} + \mathcal{O}(r^{-3})$$
$$U^A = -\frac{1}{2r^2}\mathcal{D}_B C^{BA} + \mathcal{O}(r^{-3})$$
$$\frac{V}{r} = -1 + \frac{2\, m_B(u, x^A)}{r}$$

m_B is the *Bondi mass aspect*, which corresponds to the mass of the spacetime on a null (not spacelike, like in the ADM formalism) slice, and can change with time. In fact:

$$\partial_u m_B(u, x^A) = \tfrac{1}{4}\mathcal{D}_A\mathcal{D}_B(\partial_u C^{AB}) - \tfrac{1}{8}\partial_u C^{AB}\partial_u C_{AB}$$

Simplest initial data - Schwarzschild metric

Initial data:

$$g_{AB} = r^2 \gamma_{AB}$$

we get the Schwarzschild metric in null coordinates:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m_B}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B$$

with m_B constant, i.e. $\partial_A m_B = \partial_u m_B = 0$.

This is trivially true, since imposing spherical symmetry, by Birkhoff's theorem, gives us the Schwarzschild metric, i.e. a non radiating spacetime.

BMS hair are Goldstone bosons

The variation of the tensor C_{AB} (which gives radiative spacetimes) under supertranslation is:

$$\delta_T C_{AB} = T\partial_u C_{AB} + (\gamma_{AB}\mathcal{D}^2 T + \mathcal{D}_A\mathcal{D}_B T)$$

- constant T: the tensor gets simply translated along the null coordinate $\Rightarrow$ **constant supertranslation leave the vacuum invariant**.
- non-constant T: starting with $C_{AB} = 0$ will generate a non-zero C_{AB} $\Rightarrow$ **generic supertranslations do not leave the vacuum invariant!!!**

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BMS soft-hair are the Goldstone bosons of broken supertranslation symmetry! A supertranslation transforms the Minkowski vacuum to a physically inequivalent zero-energy vacuum, with the addition (or subtraction) of a soft gravitons (and different angular momentum)

Non-zero C_{AB} initial data - Bondi-Sachs mass-loss formula

Initial data

$$g_{AB} = r^2\,\gamma_{AB} + r\,C_{AB}\ ,$$

$$C_{zz} = c\ , \quad C_{\bar{z}\bar{z}} = \bar{c}\ , \quad C_{z\bar{z}} = 0$$

we get:

$$\partial_u m_B(u, z, \bar{z}) = \partial_z^2(\partial_u \bar{c}) + \partial_{\bar{z}}^2(\partial_u c) - (\partial_u c)(\partial_u \bar{c}) \tag{0.1}$$

Integrating over the sphere, we get the **Bondi-Sachs mass-loss formula**:

$$\partial_u \int \mathrm{d}z\mathrm{d}\bar{z}\gamma_{z\bar{z}} m_B(u, z, \bar{z}) = -\int \mathrm{d}z\mathrm{d}\bar{z}\gamma_{z\bar{z}}(\partial_u c)(\partial_u \bar{c}) \tag{0.2}$$

The total energy of the system when integrated over the angles decreases, as it should.

Conservation of energy at every angle

The same formula can be integrated over the null coordinate u, instead of the angles, and gives:

$$\Delta m_B(z,\bar{z}) = -\int \mathrm{d}u(\partial_u c)(\partial_u \bar{c}) + \int \mathrm{d}u \partial_z^2(\partial_u \bar{c}) + \partial_{\bar{z}}^2(\partial_u c)$$

Strominger arXiv:1312.2229

This quantity is conserved: energy incoming from every angle on $\mathcal{I}^-$ equals the energy emerging on $\mathcal{I}^+$ at the same angle.

Global time translations = global supertranslations $\Rightarrow$ global energy conservation law

Local time translations = local (on S^2) supertranslations $\Rightarrow$ **angle by angle energy conservation law**

BMS surface charges

The covariant formalism of general relativity allows to compute the surface charges associated to supertranslations (integrability needs to be verified)

$$\begin{aligned} \mathcal{Q}_T &= \frac{1}{4\pi G}\int \mathrm{d}z\mathrm{d}\bar{z}\gamma_{z\bar{z}}\, T\,\Delta m_B(z,\bar{z}) \\ &= \frac{1}{4\pi G}\int \mathrm{d}u\Big[-T_{uu}+F^{+}\Big]\,, \qquad \text{for } T=\delta(z-w) \end{aligned}$$

with T_{uu} total outgoing radiation energy flux and F^{+} soft graviton contribution. Also crucial:

$$\mathcal{Q}_{T=1} = \frac{1}{4\pi G}\int \mathrm{d}z\mathrm{d}\bar{z}\sqrt{\gamma}\, m_B = \mathcal{H}$$

corresponds to the Hamiltonian $\mathcal{H}$ of the system . This fact is crucial to prove the BMS invariance of the quantum gravitational S-matrix.

The Strominger triangle

Strominger's papers have brought together three (at first sight) unrelated fields of gravitational physics:

- Asymptotic symmetry groups (BMS invariance of the S-matrix)

  Strominger, arXiv:1312.2229

- Weinberg's Soft theorems

  Strominger et al, arXiv:1401.7026

- Memory effects

  Strominger, Zhiboedov, arXiv:1411.5745

Invariance of the quantum gravitational S-matrix

Assume a quantum gravity S-matrix exists.
Define the BMS$^+$ ($\mathcal{I}^+$) and BMS$^-$ ($\mathcal{I}^-$) symmetries, and under certain regularity hypotheses, their diagonal subgroup **BMS^0 (i^0) is the symmetry of the S-matrix**.
To be more precise, we know that the Poisson brackets between the charges are

$$\{\mathcal{Q}_{T_1}, \mathcal{Q}_{T_2}\} = 0$$

and we also know that the $\mathcal{S}$-matrix is constructed from exponentials of the Hamiltonian, i.e. $S = \exp[f(\mathcal{H})]$ from which, trivially

$$\{\mathcal{Q}_T, \mathcal{S}\} = 0 \Leftrightarrow < \mathrm{out}|\mathcal{Q}_{T^+}\mathcal{S} - \mathcal{S}\mathcal{Q}_{T^-}|\mathrm{in} >= 0$$

or, in terms of a Ward identity in Fock space:

$$\begin{aligned} &< z_1^{out}, z_2^{out}, \dots |F^+\mathcal{S} - \mathcal{S}F^-|z_1^{in}, z_2^{in}, \cdots > \\ &\quad = \sum_k \left(E_k^{in} T(z_k^{in}) - E_k^{out} T(z_k^{out}) \right) < z_1^{out}, z_2^{out}, \dots |S|z_1^{in}, z_2^{in}, \cdots > \end{aligned}$$

This is valid in the semi-classical theory with states and operators, but not

Weinberg's soft graviton theorem

..in '65 Weinberg found a universal formula relating any $\mathcal{S}$-matrix element in a quantum theory including gravity to a second $\mathcal{S}$-matrix element which differs only by the addition of a soft graviton leg, i.e.

$$\mathcal{M}_{\mu\nu}(q, p'_1, p'_2, \dots, p_1, p_2, \dots) = \Big[\sum_{k=1}^{n} \frac{p'_{k\mu} p'_{k\nu}}{p'_k \cdot q} - \sum_{k=1}^{n} \frac{p_{k\mu} p_{k\nu}}{p_k \cdot q} \Big] \times \mathcal{M}(p'_1, p'_2, \dots, p_1, p_2, \dots)$$

for $q \to 0$

BMS invariance of S matrix = Weinberg's soft gravitons theorem

Note that:

- BMS Ward identity (BMSWI) is written in terms of boundary data in coordinate space at null infinity (radiation + soft gravitons)
- Soft gravitons theorem (SGT) : scattering of plane waves in momentum space

It can be proven that (huge abuse of notation)

$$\mathrm{BMSWI}(x) = \int exp[\mathrm{i}k \cdot x]\mathrm{SGT}(\mathrm{k}) \tag{0.3}$$

Gravitational memory effect

A finite pulse of radiation passing through a region of spacetime produce a gravitational field which induces a displacement of nearby detector (gravitational memory effect).

If one sets up ‘BMS’ detectors (i.e. they travel along world lines at fixed angle and radius), the distance shift between the detectors can be quantified in terms of supertranslation hair.

BMS hair at the horizon - the information paradox

BMS hair give reasons to doubt the assumptions of the information paradox.

- The vacuum in quantum gravity is not unique
- Black holes have infinitely many soft-hair (quantum mechanically)

Consequences of the second point: Hawking radiation will radiate supertranslation charges to null infinity. But since this charge is conserved, it must have been correlated to the charge at the horizon. This generalizes the usual energy conservation, since infinitely many of these hair exist. The process of formation/annihilation of a black hole, seen as a scattering amplitude from $\mathcal{I}^-$ to $\mathcal{I}^-$ must be constrained by the soft-graviton theorem.

Different supertranslation hair carried by a black hole, will carry different information about its formation.

Flat holography - General framework

The idea behind is to consider deformations of the known AdS/CFT correspondence. Instead of the usual $\mathrm{SO}(2, D)$ algebra of symmetries of AdS_{D+1} spacetimes and its dual conformal field theory in D dimensions, one would like to consider asymptotically flat spacetimes and find its dual field theory.
Starting points:

- Flat space asymptotic symmetries are BMS symmetries
- Flat space can be obtained as a limit of the radius of AdS $L \to \infty$

Algebraically, this corresponds to taking a contraction of the $\mathrm{SO}(2, D)$ algebra (see below)

Flat holography - 3D

The simplest case to analyze is clearly the 3 dimensional one, since the AdS_3 group of symmetries is enhanced to the infinite dimensional Virasoro group

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0}$$

BMS$_3$

Now we can follows the same procedure as before.

- Start from an ansatz for asymptotically flat spacetimes in 3D
- Find the Killing vector field that preserve its (off-shell) boundary structure
- Allow for singularities and find the algebra of vector fields
- Write down the charges algebra

BMS_3

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- Start from an ansatz for asymptotically flat spacetimes in 3D
- Find the Killing vector field that preserve its (off-shell) boundary structure
- Allow for singularities and find the algebra of vector fields
- Write down the charges algebra

The final result is:

$$
\begin{aligned}
[J_n, J_m] &= (n-m) J_{n+m} + \frac{c_1}{12} n(n^2-1)\delta_{n+m,0} \, , \\
[J_n, T_m] &= (n-m) T_{n+m} + \frac{c_2}{12} n(n^2-1)\delta_{n+m,0} \, , \\
[T_n, T_m] &= 0 \, .
\end{aligned}
$$

It is an easy exercise to show that...

Ultra-relativistic Inönü-Wigner contractions of the Virasoro algebra

..the BMS_3 algebra can be obtained as a Inönü-Wigner contraction of the Virasoro conformal algebra, by taking:

$$J_n = \lim_{\epsilon\to 0}(L_n - \bar{L}_{-n}) , \qquad T_n = \lim_{\epsilon\to 0} \epsilon(L_n + \bar{L}_{-n})$$
$$c_1 = \lim_{\epsilon\to 0}(c - \bar{c}) , \qquad c_2 = \lim_{\epsilon\to 0} \epsilon(c + \bar{c})$$

It is possible to see that this corresponds to an ultra-relativistic contraction

$$x \to x , \quad t \to \epsilon t , \quad \text{with} \quad \epsilon \to 0$$

since it sends the speed to infinity.
In the gravity language $\epsilon = \frac{1}{L}$ with L curvature of AdS.

BMS/??? in flat holography

At this point, we should find a field theory with the same ultra-relativistic algebra.
More conservatively, find a theory invariant under the Poincare' algebra of one dimension more (e.g. from the 5D flat space gravity/4D field theory perspective, the field theory needs to be invariant under the 5D Poincare' group). Then we need to enhance this to an infinite dimensional group, corresponding to BMS.
Though it is known that $\mathrm{ISO}(1, D)$ is a group contraction of $\mathrm{SO}(2, D)$, the infinite dimensional lift needs to be worked out case by case.

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Though it is known that $\mathrm{ISO}(1, D)$ is a group contraction of $\mathrm{SO}(2, D)$, the infinite dimensional lift needs to be worked out case by case.

Problematic and cumbersome, though it was worked out for the $D = 4$ case, namely flat $5D$ and $4D$ Yang Mills (or its ultra-relativistic limit)

Bagchi et al arXiv:1609.06203

Non-relativistic Inönü-Wigner contractions of the Virasoro algebra

It is quite enthralling to realize that a non-relativistic contraction of the conformal algebra in 2D, obtained by taking the opposite limit

$$x \to \epsilon x\ , \quad t \to t\ , \quad \text{with} \quad \epsilon \to 0$$

gives an algebra isomorphic to BMS_3, the GCA_2 (Galilean conformal algebra) with the generators given by:

$$J_n = \lim_{\epsilon \to 0}(L_n + \bar{L}_n)\ , \qquad T_n = \lim_{\epsilon \to 0} \epsilon(L_n - \bar{L}_n)$$
$$c_1 = \lim_{\epsilon \to 0}(c + \bar{c})\ , \qquad c_2 = \lim_{\epsilon \to 0} \epsilon(c - \bar{c})$$

Even more interestingly, the GCA_D algebra is infinite dimensional for all D

Bagchi et al, arXiv:1712.05631

BMS_3/GCA_2 correspondence

This is the formulation of flat holography that has attracted most attention, since it generalizes to every space-time dimensions and, strangely, it is a **correspondence between an ultra-relativistic theory of gravity and a non-relativistic field theory**.

Conservation of hassle: finding a gravity dual with non-relativistic symmetry algebra is a difficult problem since the metric becomes degenerate...

Regardless, one side of the correspondence, the flat gravity side is well-defined. Furthermore, 3D (super)gravity has a neat Chern-Simons formulation, which simplifies notably the computations.

Supersymmetric extensions from contractions - $\{Q, Q\} \sim P$

As a first step, we decided to find all supersymmetric extensions of the BMS_3 algebra.
The $N = 1$ case was analyzed in detail by Barnich et al, arXiv:1407.4275.

The $N = 2, \ldots, 8$ cases can be obtained by imposing a super-Poincare' subgroup inside the super-BMS group, i.e. we impose:

$$\{Q, Q\} \sim P \, , \qquad [J, Q] \sim Q$$

where P are translations and J Lorentz transformations
We find that the "correct" and *consistent contraction* of the Virasoro supercharges Q^a_r ($\bar{Q}^a_r$) is the democratic contraction:

$$\mathcal{Q}^{1,a}_r = \lim_{\epsilon \to 0} \sqrt{\epsilon}\, Q^a_r \, , \qquad \mathcal{Q}^{2,a}_r = \lim_{\epsilon \to 0} \sqrt{\epsilon} \, , \bar{Q}^a_{-r}$$

where $a = 1, \ldots, N/2$.

N=2 Super-BMS$_3$ algebra

The full algebra reads:

$$\begin{aligned}
[J_n, J_m] &= (n-m)J_{n+m} + \frac{c_1}{12}n(n^2-1)\delta_{n+m,0} \ , \\
[J_n, T_m] &= (n-m)T_{n+m} + \frac{c_2}{12}n(n^2-1)\delta_{n+m,0} \ , \\
[T_n, T_m] &= 0 \ , \qquad\qquad [T_n, Q_r] = 0 \ , \\
[J_n, \mathcal{Q}_r] &= \left(\frac{n}{2} - r\right)\mathcal{Q}_{r+n} \ , \\
\{\mathcal{Q}^a_r, \mathcal{Q}^b_s\} &= \frac{1}{2}\delta^{ab}\left[T_{r+s} + \frac{c_2}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\right] .
\end{aligned}$$

Banerjee, Jatkar, IL, Mukhi, Neogi arXiv:1609.09210

Super-BMS$_3$ algebra

Of course, the algebra(s) we obtained from the connection with the Virasoro algebra can also be derived by a proper analysis of the precise boundary conditions to impose on the gravitational phase space (as we did before).

There are many subtleties, mostly related to the presence of the R-symmetry in the $N = (n_1, n_2)$ super-Virasoro algebra with $n_{1,2} \geq 2$

IL, Merbis arXiv:1610.07506
Banerjee, IL, Neogi arXiv:1706.02922
Banerjee, IL, Neogi to be submitted

Other consistent supersymmetrizations

We can weaken our initial hypotheses, by imposing only consistency of the contraction (hence not requiring that the super-Poincare subgroup of super-BMS$_3$ corresponds to the usual super-Poincare' algebra).

In this case, we find some curious, novel, result for the supersymmetry algebra. Specifically, consider the following despotic contraction of the Virasoro generators:

$$R_r = \lim_{\epsilon \to 0} \epsilon (Q_r - \bar{Q}_r) \;, \qquad G_r = \lim_{\epsilon \to 0} (Q_r + \bar{Q}_r)$$

This is a non-relativistic limit, and is consistent only if we start from an anti-unitary representation of the barred Virasoro generators, i.e.

$$\bar{Q}_r^\dagger = -\bar{Q}_{-r} \;, \qquad Q_r^\dagger = Q_{-r}$$

from which we get immediately:

$$R_r^\dagger = R_{-r} \;, \qquad G_r^\dagger = G_{-r} \tag{0.4}$$

The despotic algebra - Exotic energy bounds

The supersymmetry charge algebra stemming from this contraction reads:

$$[\mathcal{J}_n, \mathcal{G}_r] = \left(\frac{n}{2} - r\right)\mathcal{G}_{n+r} \ , \qquad [\mathcal{J}_n, \mathcal{R}_r] = \left(\frac{n}{2} - r\right)\mathcal{R}_{n+r} = [\mathcal{M}_n, \mathcal{G}_r]$$

$$\{\mathcal{G}_r, \mathcal{G}_s\} = \mathcal{J}_{r+s} + \frac{c_1}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}$$

$$\{\mathcal{R}_r, \mathcal{G}_s\} = \mathcal{M}_{r+s} + \frac{c_2}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \ , \qquad \{\mathcal{R}_r, \mathcal{R}_s\} = 0 \ .$$

from which:

$$\mathcal{J}_0 = \mathcal{G}_{1/2}\mathcal{G}_{-1/2} + \mathcal{G}_{-1/2}\mathcal{G}_{1/2} \geq 0 \ ,$$
$$-\frac{1}{2}\mathcal{J}_0 \leq \mathcal{M}_0 \leq \frac{1}{2}\mathcal{J}_0$$

These are 'novel' energy bounds which need to be studied more in detail.

IL, Merbis arXiv:1610.07506

Conclusions

Asymptotic symmetry algebras are an extremely fertile ground of investigations.
Aside from what shown, there are numerous other applications of interest for

- The geometric structure of GR
- Black holes physics
- Conformal field theories and their ultra- and non-relativistic limits
- BMS (and GCA) invariant theories - bootstraps
- ...

IL, Merbis, Zondinmawia to be submitted

THANK YOU FOR YOUR ATTENTION!

BMS on m_B and C_{AB}

The metric is:

$$\mathrm{d}s^2 = \Big(-1 + \frac{2m_B}{r} \Big) \mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{4r^2}{(1+z\bar{z})^2} + rC_{zz}\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\mathrm{d}\bar{z}^2$$
$$- 2U_z \mathrm{d}u\mathrm{d}z - 2U_{\bar{z}}\mathrm{d}u\mathrm{d}\bar{z}$$

$$U_z = -\tfrac{1}{2} D^z C_{zz} \; , \qquad U_{\bar{z}} = -\tfrac{1}{2} D^{\bar{z}} C_{\bar{z}\bar{z}}$$

The Killing vector of BMS transformation is:

$$\xi = T\partial_u - \frac{1}{r}(D^z T \partial_z + D^{\bar{z}} T \partial_{\bar{z}}) + D^z D_z T \partial_r$$

The Lie derivative of the metric reads

$$\delta_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\mu\alpha} \qquad \Rightarrow$$

BMS on m_B and C_{AB}

In the limit $r \to \infty$, we can write

$$\delta_\xi g_{uu} = \delta_\xi(\frac{2m_B}{r}) = T\partial_u(\frac{2m_B}{r}) + \mathcal{O}(r^{-2}) \Rightarrow$$
$$\delta_T m_B = T\partial_u m_B \ ,$$

$$\begin{aligned}\delta_\xi g_{zz} = \delta_\xi(rC_{zz}) &= T\partial_u(rC_{zz}) - 2r(\partial_z D^{\bar{z}} T)\gamma_{z\bar{z}} + \mathcal{O}(1) \\ &= T\partial_u(rC_{zz}) - 2r(D_z D^{\bar{z}} T)\gamma_{z\bar{z}} + \mathcal{O}(1) \Rightarrow\end{aligned}$$
$$\delta_T C_{zz} = T\partial_u C_{zz} - 2D_z D_z T \ ,$$

where we used the fact that $\Gamma^{\bar{z}}_{z,\alpha} \sim \frac{1}{r}$ and $\gamma_{z\bar{z}} D^{\bar{z}} = D_z$.

Exotic energy bounds - despotic N=2 super-BMS$_3$

Start from the algebra relations:

$$\{\mathcal{G}_r, \mathcal{G}_s\} = \mathcal{J}_{r+s} + \frac{c_1}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \ , \{\mathcal{R}_r, \mathcal{R}_s\} = 0$$
$$\{\mathcal{R}_r, \mathcal{G}_s\} = \mathcal{M}_{r+s} + \frac{c_2}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \ .$$

From these algebra relation it is easy to show that:

$$\mathcal{J}_0 = \mathcal{G}_{1/2}\mathcal{G}_{-1/2} + \mathcal{G}_{-1/2}\mathcal{G}_{1/2} \geq 0 \ ,$$
$$\mathcal{M}_0 = \tfrac{1}{2}\left[\mathcal{R}_{1/2}\mathcal{G}_{-1/2} + \mathcal{G}_{-1/2}\mathcal{R}_{1/2} + \mathcal{R}_{-1/2}\mathcal{G}_{1/2} + \mathcal{G}_{1/2}\mathcal{R}_{-1/2}\right]$$

We can also take the combinations:

$$q_r = \tfrac{1}{\sqrt{2}}(\mathcal{G}_r + \mathcal{R}_r) \ , \quad \tilde{q}_r = \tfrac{1}{\sqrt{2}}(\mathcal{G}_r - \mathcal{R}_r) \ , \quad q_r^\dagger = q_{-r} \ , \tilde{q}_r^\dagger = q_{-r}$$

Exotic energy bounds - despotic N=2 super-BMS$_3$

It is easy to show now that:

$$0 \leq q_{1/2}q^{\dagger}_{1/2} + q^{\dagger}_{1/2}q_{1/2} = \tfrac{1}{2}\mathcal{J}_0 + \mathcal{M}_0 \Rightarrow \mathcal{M}_0 \geq -\tfrac{1}{2}\mathcal{J}_0$$
$$0 \leq \tilde{q}_{1/2}\tilde{q}^{\dagger}_{1/2} + \tilde{q}^{\dagger}_{1/2}\tilde{q}_{1/2} = \tfrac{1}{2}\mathcal{J}_0 - \mathcal{M}_0 \Rightarrow \mathcal{M}_0 \leq \tfrac{1}{2}\mathcal{J}_0 \Rightarrow$$

and the two conditions need to be satisfied simultaneously. Hence

$$-\tfrac{1}{2}\mathcal{J}_0 \leq \mathcal{M}_0 \leq \tfrac{1}{2}\mathcal{J}_0$$

as we wanted to show

Some Killing vectors equations

$$\mathcal{L}_\xi g_{rA} = 0 \Rightarrow 2\nabla_{(r}\xi_{A)} = g_{AB}(\nabla_r \xi^B + g^{BC} g_{ur} \nabla_C \xi^u) = 0 \Rightarrow$$

$$\partial_r \xi^B = g^{BC} e^{2\beta} \partial_C f \Rightarrow \xi^A = Y^A(u, x^B) - \partial_B f \int_r^\infty \mathrm{d}r' e^{2\beta} g^{AB} \ ,$$